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ON APPROXIMATE METHODS OF ANALYSING CERTAIN SINGULARLY-PERTURBED SYSTEMS*

L.K. KUZ'MINA

A certain class of singularly-perturbed systems which have a variety of m-dimensional stationary positions is considered. When a small parameter disappears, the system also has an m-dimensional manifold of stationary positions and, therefore, the corresponding characteristic equation has m zero roots. The conditions under which the solution of a stability problem reduces to the same problem for a degenerate system are defined. As an application in practice gyroscopic stabilizing systems (the critical case corresponds to such systems) with elastic elements of high stiffness are discussed. The conditions under which the solution of the problem of the stability of steady motion follows from the solution of this problem for an ideal system (with absolutely rigid elements) are obtained. The problem of the closeness of the corresponding solutions of the complete and a simplified system of differential equations over an infinite time interval is discussed.

1. Suppose the perturbed motion of a system is described by a differential equation of the form

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$$\frac{d\mathbf{z}}{dt}=\mathbf{Z}(t,\mu,\mathbf{z},\mathbf{x}),\quad \frac{d\mathbf{x}_1}{dt}=P_1(\mu)\mathbf{x}+\mathbf{X}_1(t,\mu,\mathbf{z},\mathbf{x}) \tag{1.1}$$
$$\mu^2\frac{d\mathbf{x}_2}{dt}=P_2(\mu)\mathbf{x}+\mathbf{X}_2(t,\mu,\mathbf{z},\mathbf{x}),\quad \frac{d\mathbf{x}_3}{dt}=P_3(\mu)\mathbf{x}+\mathbf{X}_3(t,\mu,\mathbf{z},\mathbf{x})$$
$$(\mathbf{x}=\|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\|^T,\ P_i(\mu)=\|P_{i1}(\mu),P_{i2}(\mu),\ P_{i3}(\mu)\|)$$

Here $\mathbf{z}$ is an m-dimensional vector which corresponds to the Lyapunov-critical variables, $\mathbf{x}_i$ $(i=1,2,3)$ are n_i-dimensional vectors corresponding to the non-critical variables, the index 'T' denotes transposition, μ is a small positive parameter, $P_i(\mu)$ are matrices of corresponding dimensions whose elements are continuous functions of μ, $P_{ij}(\mu)$ are $n_i\times n_j$ submatrices $(i,j=1,2,3)$, $\mathbf{Z}$ and $\mathbf{X}_i$ are certain functions which are holomorphic (in the corresponding domain) with respect to the aggregate of variables $\mathbf{z}$ and $\mathbf{x}$, they do not contain in their expansions any terms below the second dimension, and their coefficients are continuous bounded functions of t and μ.

Let $\mathbf{Z},\mathbf{X}_i$ $(i=1,2,3)$ vanish for $\mathbf{x}=0$; and let $P_{21}(\mu)=\mu P_{21}'(\mu)$, $P_{22}(\mu)=\mu P_{22}'(\mu)$.

The characteristic equation for system (1.1) has m zero roots. We will denote the equation which determines the remaining roots by

$$d(\lambda,\ \mu)=0 \tag{1.2}$$

without writing it in detail.

Putting $\mu=0$, we obtain from (1.1) the degenerate system

$$d\mathbf{z}/dt=\mathbf{Z},\quad d\mathbf{x}_1/dt=P_1\mathbf{x}+\mathbf{X}_1 \tag{1.3}$$
$$0=P_{23}\mathbf{x}_3+\mathbf{X}_2,\quad d\mathbf{x}_3/dt=P_3\mathbf{x}+\mathbf{X}_3$$
$$(\mathbf{z}=\mathbf{z}(t,0,\mathbf{Z},\mathbf{X}),\ \mathbf{X}_i=\mathbf{X}_i(t,0,\mathbf{z},\mathbf{x})\ (i=1,2,3)$$
$$P_1=P_1(0),\ P_{23}=P_{23}(0),\ \ P_3=P_3(0))$$

Assuming that we are dealing with specific mechanical systems, we consider the case where $n_2=n_3$, and at the same time the adiabatic equation $0=P_{23}\mathbf{x}_3+\mathbf{X}_2$ from (1.3) admits of the unique solution $\mathbf{x}_3=0$, and the equation $0=P_{31}\mathbf{x}_1+P_{32}\mathbf{x}_2+\mathbf{X}_3(t,0,\mathbf{z},\mathbf{x}_1,\mathbf{x}_2,0)$ has the solution $\mathbf{x}_2=\mathbf{f}(t,\mathbf{z},\mathbf{x}_1)$. On substituting $\mathbf{x}_3=0$ and $\mathbf{x}_2=\mathbf{f}(t,\mathbf{z},\mathbf{x}_1)$ into the first two equations of system (1.3), we obtain

$$d\mathbf{z}/dt=\mathbf{Z}'(t,\mathbf{z},\mathbf{x}_1),\quad d\mathbf{x}_1/dt=P_{11}'\mathbf{x}_1+\mathbf{X}_1'(t,\mathbf{z},\mathbf{x}_1) \tag{1.4}$$

We take this system as a simplified version of (1.1). The characteristic equation of the shortened system has m zero roots, and the remaining roots are found from the equation

$$|\lambda E-P_{11}'|=0\quad (P_{11}'=P_{11}-P_{12}P_{32}^{-1}P_{31}) \tag{1.5}$$

System (1.4) is of a lower order than the total system (1.1), and the following important problem arises from the practical point of view: under what conditions, for a sufficiently small parameter μ, does the stability of the zero solution of the simplified system (1.4) imply the stability of the zero solution of the complete system (1.1)? A similar problem for differential equations with a small parameter in derivative terms has been discussed by many authors, for example in /1–5/, for cases different from the one discussed here.

Taking into account /4, 5/ and making use of the corresponding Lyapunov theorems, one can show that for system (1.1) the following assertion is valid.

Theorem 1. If for

$$|P_{23}|\neq 0,\quad |P_{32}|\neq 0,\quad \begin{vmatrix}P_{11}, & P_{12}\\ P_{31}, & P_{32}\end{vmatrix}\neq 0$$

the equation

$$\begin{vmatrix}\alpha E-P_{22}', & -P_{23}\\ -P_{32}, & \alpha E\end{vmatrix}=0 \tag{1.6}$$

satisfies the Hurwitz condition, and all roots, with the exception of m zero roots of the characteristic equation of the simplified system, have negative real parts, then the zero solution of system (1.4) is stable, and for a sufficiently small parameter μ, the zero solution of the complete system (1.1) is stable as well.

Here the simplified system (1.4) has an integral of the form $\mathbf{z}+\boldsymbol{\varphi}(t,\mathbf{z},\mathbf{x}_1)=\mathbf{B}$, and the complete system admits of the integral $\mathbf{z}+\mathbf{F}(t,\mu,\mathbf{z},\mathbf{x})=\mathbf{A}$ where $\boldsymbol{\varphi}$ and $\mathbf{F}$ are non-linear holomorphic vector functions, with $\boldsymbol{\varphi}=0$ for $\mathbf{x}_1=0$, and $\mathbf{F}=0$ for $\mathbf{x}=0$, $\mathbf{B}$ and $\mathbf{A}$ being constant vectors.

Under the conditions of the theorem, any solution of the form $\mathbf{z}=\mathbf{C}$, $\mathbf{x}=0$ ($\mathbf{C}$ is an arbitrary vector) will also be stable; here $\|\mathbf{C}\|$ is a fairly small quantity.

2. As an application of the results obtained, we consider the stability of the sustained motion of a gyrostabilization system where a case which is critical (according to Lyapunov) arises under the assumption that the system's elements are not absolutely rigid. Such a system was discussed in /5/ where a simplified model was introduced with the additional condition that the gyroscopes have their own angular momenta.

Here we shall consier a gyrostabilized system, assuming that the rigidity of its elastic elements is sufficiently high (but finite in contrast to the dieal, that is an absolutely rigid system). For simplicity, we assume that the electric circuits of the tracking systems are delayless, and the differential equations of the perturbed motion (see /5/) are reduced to the form

$$\frac{d}{dt} a\mathbf{q}_M^{\cdot} + (b^\circ + g^\circ)\mathbf{q}_M^{\cdot} + c^\circ \mathbf{q}_M = \mathbf{Q}_M'', \quad \frac{d\mathbf{q}_M}{dt} = \mathbf{q}_M^{\cdot} \tag{2.1}$$

As in /5/, here $\mathbf{q}_M$ is the n-dimensional vector of the generalized mechanical coordinates, $\mathbf{q}_M = \| \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4 \|^T$, where $\mathbf{q}_1$ is the l-dimensional vector of the angles of precession of the gyroscopes, $\mathbf{q}_2$ is the $(m-l)$-dimensional vector of the deviations of the angles of the gyroscopes' natural rotations from their steady-motion values, $\mathbf{q}_3$ is the $(s-m)$-dimensional vector of the angles of rotation of the rotors of the stabilizing motors $(s = m + l)$, $\mathbf{q}_4$ is the $(n-s)$-dimensional deformation vector of the elastic elements, a, b and g are square $(n \times n)$-matrices of the quadratic form of the system's kinetic energy, the dissipation function of the friction forces, and the gyroscopic coefficient respectively, $c = \| 0, 0, c_3, c_4 \|^T$, with $c_3 = \| c_{31}, 0, 0, 0 \|$, $c_4 = \| 0, 0, 0, c_{44} \|$, c_{44} is a square $(n-s) \times (n-s)$-matrix which corresponds to the potential energy of the elasticity forces, $b = \| b_1, b_2, b_3, b_4 \|^T$ with $b_i = \| b_{i1}, b_{i2}, b_{i3}, b_{i4} \|$ $(i = 1, 2, 3, 4)$, where b_{ij} are submatrices of the corresponding dimensions, b_{44} is a square $(n-s) \times (n-s)$-matrix of the dissipation function of the internal friction forces in the material of solids, and $g = \| g_1, g_2, g_3, g_4 \|^T$, $g_i = \| g_{i1}, g_{i2}, g_{i3}, g_{i4} \|$ $(i = 1, 2, 3, 4)$, where g_{ij} are submatrices of corresponding dimensions of the matrix g. The circles denote the zero-order terms in the expansions of the corresponding functions.

Consider systems with fairly stiff elements. Let $c_{44} = c_{44}^* / \mu^2$, $b_{44} = b_{44}^* / \mu$, where μ is a small dimensionless positive parameter. A model of this kind was discussed in /7/.

Let us reduce system (2.1) to the form (1.1). We introduce new variables

$$\mathbf{z} = \left\| \begin{matrix} a_1 \\ a_2 \end{matrix} \right\| \mathbf{q}_M^{\cdot} + \left\| \begin{matrix} b_1^\circ + g_1^\circ \\ b_2^\circ + g_2^\circ \end{matrix} \right\| \mathbf{q}_M, \quad \boldsymbol{\varkappa}_1 = \left\| \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} \right\| \mathbf{q}_M^{\cdot}$$

$$\boldsymbol{\varkappa}_2 = a_4 \mathbf{q}_M^{\cdot}, \quad \mathbf{q}_1 = \mathbf{q}_1, \quad \mathbf{q}_4 = \mathbf{q}_4$$

where $\mathbf{z}$, $\boldsymbol{\varkappa}_1$, $\boldsymbol{\varkappa}_2$ are m-, s-, and $(n-s)$-dimensional vectors, and a_i $(i = 1, 2, 3, 4)$, b_j and g_j $(j = 1, 2)$ are submatrices, of corresponding dimensions of the matrices a, b and g respectively.

The above transformation is non-linear, and non-singular, and under the condition

$$\begin{vmatrix} b_{12}^\circ - g_{12}^\circ, & b_{13}^\circ + g_{13}^\circ \\ b_{22}^\circ + g_{22}^\circ, & b_{23}^\circ + g_{23}^\circ \end{vmatrix} \neq 0 \tag{2.2}$$

is uniformly regular and does not change the formulation of the stability problem.

In the new variables, system (2.1) becomes

$$\frac{d\mathbf{z}}{dt} = \mathbf{Z}, \quad \frac{d\boldsymbol{\varkappa}_1}{dt} = -e_{11}^\circ \boldsymbol{\varkappa}_1 - e_{12}^\circ \boldsymbol{\varkappa}_2 + \left\| \begin{matrix} 0 \\ -c_{31}^\circ \end{matrix} \right\| \mathbf{q}_1 + \mathbf{K}_1 \tag{2.3}$$

$$\mu^2 \frac{d\boldsymbol{\varkappa}_2}{dt} = -e_{21}^\circ \boldsymbol{\varkappa}_1 - e_{22}^\circ \boldsymbol{\varkappa}_2 - c_{44}^{*\circ} \mathbf{q}_4 + \mathbf{K}_2$$

$$\frac{d\mathbf{q}_1}{dt} = l_{11}\boldsymbol{\varkappa}_1 + l_{12}\boldsymbol{\varkappa}_2, \quad \frac{d\mathbf{q}_4}{dt} = l_{41}\boldsymbol{\varkappa}_1 + l_{42}\boldsymbol{\varkappa}_2$$

Here e_{ij} $(i, j = 1, 2)$ are submatrices, of corresponding dimensions, of the matrix $e = (b(\mu) + g(\mu)) d$, $d = a^{-1} = \| d_1, d_2, d_3, d_4 \|^T$, $d_i = \| l_{i1}, l_{i2} \|$ $(i = 1, 2, 3, 4)$, d_i and l_{ij} are submatrices, of corresponding dimensions of the matrix d, $\mathbf{z}$ are critical variables, $\boldsymbol{\varkappa}_1, \boldsymbol{\varkappa}_2, \mathbf{q}_1, \mathbf{q}_4$ are non-critical variables, $\mathbf{Z}, \mathbf{K}_1, \mathbf{K}_2$ denote non-linear vector functions holomorphic with respect to the aggregate of the variables $\mathbf{z}, \boldsymbol{\varkappa}_1, \boldsymbol{\varkappa}_2, \mathbf{q}_1, \mathbf{q}_4$, which vanish for zero values of the non-critical variables.

Notice that system (2.3) has the form

$$d\mathbf{z}/dt = \mathbf{Z}, \quad d\mathbf{x}_1/dt = P_1\mathbf{x} + \mathbf{X}_1 \tag{2.3'}$$

$$\mu^2 \frac{d\mathbf{x}_2}{dt} = P_2(\mu)\mathbf{x} + \mathbf{X}_2(\mu, \mathbf{z}, \mathbf{x}), \quad \frac{d\mathbf{x}_3}{dt} = P_3\mathbf{x} + \mathbf{X}_3$$

Here

$$\mathbf{x}_1 = \left\| \begin{matrix} \boldsymbol{\varkappa}_1 \\ \mathbf{q}_1 \end{matrix} \right\|, \quad \mathbf{x}_2 = \boldsymbol{\varkappa}_2, \quad \mathbf{x}_3 = \mathbf{q}_4$$

$$P_{11}=\left\|\begin{array}{c|c} -e_{11}^\circ & \begin{array}{c}0\\ \hline -c_{31}^\circ\end{array} \\ \hline l_{11}^\circ & 0\end{array}\right\|,\quad P_{12}=\left\|\begin{array}{c}-e_{12}^\circ\\ \hline l_{12}^\circ\end{array}\right\|,\quad P_{13}=0$$

$$P_{21}(\mu)=\|-e_{21}^\circ, 0\|,\quad P_{22}(\mu)=-e_{22}^\circ,\quad P_{23}=-c_{44}^{*\circ}$$

$$P_{31}=\|l_{41}^\circ, 0\|,\quad P_{32}=l_{42}^\circ,\quad P_{33}=0$$

$$P_{21}(\mu)=\mu P_{21}'(\mu),\quad P_{22}(\mu)=\mu P_{22}'(\mu)$$

On subsituting $\mu=0$ we obtain a degenerate system. At the same time, taking into accout the fact that for the systems discussed $|c_{44}^*|\neq 0$, $|l_{42}|\neq 0$, taking $q_4=0$, $\varkappa_2=-l_{42}^{-1}l_{41}\varkappa_1$ and subsituting these expressions into the first two equations of (2.3′), we can represent this system in the form

$$\frac{dz}{dt}=Z',\quad \frac{d\varkappa_1}{dt}=e_{11}'\varkappa_1+\left\|\begin{array}{c}0\\ -c_{31}^\circ\end{array}\right\|q_1+K_1',\quad \frac{dq_{-1}}{dt}=l_{11}'\varkappa_1 \tag{2.4}$$

(the primes denote the corresponding transformed matrices and vector functions obtained as a result of the substitution).

We take (2.4) as a simplified system for (2.3). In the old variables the system

$$\frac{d}{dt}a^*q^\cdot+(b^*+g^*)q^\cdot+c^*q=Q^*,\quad \frac{dq}{dt}=q^\cdot \tag{2.5}$$

corresponds to (2.4). Here $q=\|q_1, q_2, q_3\|^T$ is an s-dimensional vector of the generalized coordinates which describe the state of the absolutely stiff system, and a^*, b^*, g^*, c^* are square $(s\times s)$-matrices corresponding to an absolutely stiff mechanical system.

We note that the differential Eq.(2.5), which form a system of lower order compared with (2.1), describe the motion of a mechanical system whose elements possess absolute rigidity.

We pose the following problem: under what conditions is it permissible to use Eq.(2.5) instead of the initial Eq.(2.1) (that is, we allow a transition from the model of a mechanical system which takes into account the real properties, to its idealized (absolutely rigid) model)?

Following the results obtained in Sect.1, it can be shown that if for

$$\left\|\begin{array}{c|c} -e_{11}^\circ,\ -e_{12}^\circ & \begin{array}{c}0\\ \hline -c_{31}^\circ\end{array} \\ \hline \begin{array}{c}d_1^\circ\\ d_4^\circ\end{array} & 0\end{array}\right\|\neq 0$$

the equation

$$\left|\begin{array}{cc}\alpha E+e_{22}', & c_{44}^{*\circ}\\ -l_{42}^\circ, & \alpha E\end{array}\right|=0 \tag{2.6}$$

satisfies the Hurwitz condition, and the characteristic equations of the linear approximation system for (2.4) have all their roots (with the exception of m zero roots) in the left half-plane, then for sufficiently small μ the stability of a zero solution of the simplified system (2.4) secures the stability of the zero solution of the complete system (2.3) as well. Turning to the old variables we note that the equation

$$\left|\begin{array}{c|c}\multicolumn{2}{c}{a_I^\circ}\\ \hline a_{II}^\circ\alpha^2 & a_{44}^\circ\alpha^2+b_{44}^{*\circ}\alpha+c_{44}^{*\circ}\end{array}\right|=0 \tag{2.7}$$

$$(a_I=\|a_1, a_2, a_3\|^T,\ a_{II}=\|a_{41}, a_{42}, a_{43}\|)$$

corresponds to Eq.(2.6). Eq.(2.7) satisfies, as shown in /7/, for the mechanical systems discussed, the Hurwitz condition for any value of the system's parameters with physical meaning (here a_{4i} $(i=1, 2, 3, 4)$ are submatrices, of corresponding dimensions. of the matrix a). This proves the theorem.

Theorem 2. If, under condition (2.2) and for $|c_{31}^\circ|\neq 0$, all roots (with the exception of m zero roots) of the characteristic equations of the simplified system (2.5) have negative real parts, then the zero solution of this system is stable, and for a sufficiently small parameter μ (with sufficiently high rigidity of the system's elements) the zero solution of the complete system (2.1) is stable as well.

The simplified system (2.5) admits of an integral of the form

$$\left\|\begin{array}{c}a_1^*\\ a_2^*\end{array}\right\|q^\cdot+\left\|\begin{array}{c}b_1^{*\circ}+g_1^{*\circ}\\ b_2^{*\circ}+g_2^{*\circ}\end{array}\right\|q+\varphi(q, q^\cdot)=B$$

and the complete system an integral of the form

$$\begin{Vmatrix} a_1 \\ a_2 \end{Vmatrix} \mathbf{q}_M^{\cdot} + \begin{Vmatrix} b_1^{\circ} + g_1^{\circ} \\ b_2^{\circ} + g_2^{\circ} \end{Vmatrix} \mathbf{q}_M + \mathbf{F}(\mathbf{q}_M, \mathbf{q}_M^{\cdot}) = \mathbf{A}$$

Note. 1°. Theorem 2 has been proved for gyrostabilizing systems, but all the operations are easily extended to the general case of mechanical systems without gyroscopes, for which an analogous assertion holds.

Note. 2°. Theorem 2 refers to the case of non-asymptotic stability of the zero solution, but a similar result is obtained when the zero solution is asymptotic.

The results obtained define the conditions which make it possible, in solving the stability problem for the systems discussed, to use the simplified model of a lower order corresponds to an absolutely rigid mechanical system.

3. For the mechanical systems discussed in Sect.2, we pose the problem of the closeness of the solutions of the complete and simplified systems of differential equations. Let $\mathbf{q}_i = \mathbf{q}_i(t, \mu)$, $\mathbf{q}_i^{\cdot} = \mathbf{q}_i^{\cdot}(t, \mu) (i = 1, 2, 3, 4)$ be a solution of system (2.1) with the initial condition $\mathbf{q}_{i0} = \mathbf{q}_i(t_0, \mu)$, $\mathbf{q}_{i0}^{\cdot} = \mathbf{q}_i^{\cdot}(t_0, \mu)$; we denote the solution of the simplified system (2.5) by $\mathbf{q}_i^* = \mathbf{q}_i^*(t)$, $\mathbf{q}_i^{\cdot *} = \mathbf{q}_i^{\cdot *}(t)$ $(i = 1, 2, 3, 4)$ which is determined by the initial conditions $\mathbf{q}_{i0}^* = \mathbf{q}_i^*(t_0)$, $\mathbf{q}_{i0}^{\cdot *} = \mathbf{q}_i^{\cdot *}(t_0) (i = 1, 2, 3)$, $\mathbf{q}_4^* = 0$, $\mathbf{q}_4^{\cdot *} = 0$.

Let us find the conditions under which the corresponding solutions of the complete and simplified systems are close in an infinite time interval. Applying the methods of the theory of stability in combination with that of singularly-perturbed equations (see /8—10/), we consider the differential equations for the deflections which correspond to the non-critical variables, use the integrals which occur for the systems in question, and allow for the special features of these systems given in Sect.2. Then we can show that the following assertion holds.

Theorem 3. If condition (2.2) is satisfied, and if $|c_{31}^{c}| \neq 0$ and all roots (with the exception of *m* zero roots) of the characteristic equation for the simplified system (2.5) are in the left half-plane, then for a sufficiently small parameter μ, for the previously specified numbers $\xi > 0$ and $\eta > 0$ (where ξ is as small as desired) there exists μ_* such, that in perturbed motion when $0 < \mu < \mu_*$ for all $t > t_* > t_0$, the relations

$$\| \mathbf{q}_i - \mathbf{q}_i^* \| < \xi, \quad \| \mathbf{q}_i^{\cdot} = \mathbf{q}_i^{\cdot *} \| < \xi \; (i = 1, 2, 3, 4)$$

are satisified if $\mathbf{q}_{j0} = \mathbf{q}_{j0}^*$, $\mathbf{q}_{j0}^{\cdot} = \mathbf{q}_{j0}^{\cdot *}$ $(j = 1, 2, 3)$, $\| \mathbf{q}_{40} \| < \eta$, $\| \mathbf{q}_{40}^{\cdot} \| < \eta$. Here by selecting sufficiently small μ, the value of t_* can be made as close to t_0 as desired.

Note that many papers have been devoted to various problems of the dynamics of mechanical systems with elastic elements of high stiffness; for example, the questions of constructing asymptotic forms of the solutions of differential equations (see /11, 12/), or the stability problems for specific types of gyrostabilizers, /13, 14/, etc. As mentioned above, for the gyroscopic systems discussed in the present paper, no assumption regarding the large magnitude of the angular momenta of gyroscopes is made (unlike in /5/), and the equation of precession theory, which has been studied by many authors (for example, in /14—19/ etc.), are not considered here. The results obtained in the present paper complement previous studies, substantiate, for the system discussed above, the feasibility of using the simplified model employed, and make it possible to use the approximate equations and solution in dealing with the problems in question.

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ON THE STABILITY OF THE VERTICAL ROTATION OF A SOLID SUSPENDED ON A ROD*

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The problem of the motion of a dynamically symmetric solid suspended from a fixed point by a weightless rod and two ball and socket joints one of which is fixed at the fixed point O', and the other is on the body axis of symmetry at the point O is considered. The question of the stability of the uniform body rotation when points O' and O, and the body centre of inertia C lie on the same vertical, and at the same time point O may be either above or below point O', and point C either above or below point O, is discussed. An analysis of the necessary and sufficient conditions for stability is given. The set of all the system's parameters is reduced to three independent dimensionless parameters L, Ω and β, and in the plane (L, Ω), for fixed values of β, the regions for which the unperturbed rotation is steady, or steady to a first approximation, or non-steady are indicated. The regions for which the body rotation is steady to a first approximation when the point O is situated higher than the point O', and the point C lies higher or lower than the point O are determined.

The sufficient conditions for the vertical rotation of a dynamically symmetric body suspended on a filament were obtained in /1/ and investigated for the cases where in non-perturbed motion the point C is below point O, when points C and O coincide, and when the length of the filament is zero (Lagrange gyroscope). In /2/ an analysis is given of the sufficient conditions for stability obtained in /1/, and also the necessary conditions for the cases where in a non-perturbed motion point C is located above point O.

1. Consider, in a uniform field of gravity, the motion of a dynamically symmetric solid suspended on a thin straight weightless rod and two ball and socket joints, one of them being the fixed point O', and the other located on the axis of symmetry of the body at point O.

We adopt the coordinate system $Ox_1x_2x_3$ whose axes are invariably linked with the body and directed along its principal axes of inertia for the point O. Let us introduce the following notation: m, $\mathbf{J}_C$ is the mass and the tensor of inertia of the body for its centre of mass C, with the diagonal elements $J_1 = J_2$, J_3; $\boldsymbol{\omega}$, $\mathbf{K}_C = \mathbf{J}_C \cdot \boldsymbol{\omega}$, are the angular velocity and the momentum of the body, computed for point C, $\mathbf{a}$ is the radius vector of point C relatively to point O, $\mathbf{v}$ is the velocity of point O, $\boldsymbol{\gamma}$ is the unit vector of the upward vertical, l is the length of the rod, $\mathbf{e}$ is the unit vector directed along the rod to point O', g is the acceleration due to gravity, and N is the reaction of the rod. We shall express all vectors by their projections ω_i, $K_{Ci} = J_i\omega_i$, v_i, γ_i, e_i, a_i on the x_i axis $(i = 1, 2, 3)$ with $a_1 = a_2 = 0$, $a_3 = a$.

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